Indian Statistical Institute, Bangalore
M. Math.
Second Year, Second Semester
Advanced Linear Algebra

Back paper examination — Date : June 6, 2025
Total Marks: 100 — Time: 3 hours
Maximum marks: 100 — Instructor: B V Rajarama Bhat

(1) Suppose A is a $n \times n$ doubly stochastic matrix and $x \in \mathbb{R}^n$. Show that Ax is majorized by x. Conversely, show that if B is a real $n \times n$ matrix such that By is majorized by y for every $y \in \mathbb{R}^n$, then B is doubly stochastic. [20]

(2) Let $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ be a linear functional defined by

$$\varphi(X) = \text{ trace } (\rho X), \ \forall X \in M_n(\mathbb{C}),$$

for some matrix $\rho \in M_n(\mathbb{C})$. Show that φ is a positive linear functional if and only if ρ is positive. [15]

(3) Let H be a graph with $V(H) = \{1, 2, 3, 4\}$

$$E(H) = \{\{i, j\} : i \neq j, \{i, j\} \neq \{1, 2\}, \{i, j\} \neq \{1, 3\}, i, j \in V(H)\}.$$

(i) Describe/draw all the spanning trees of H. (ii) Verify the matrix-tree theorem for this example. [15]

(4) (Line graph). Let G be a graph with at least one edge. Then the line graph of G is the graph $\hat{G}$, where $V(\hat{G}) = E(G)$ (So the vertex set of $\hat{G}$ is the set of edges of G) and two edges of G form an edge in $\hat{G}$ if the edges have a common incident vertex. (i) If N is the incidence matrix of G, show that $N^t N - 2I$ is the adjacency matrix of $\hat{G}$. (ii) If λ is an eigenvalue of the adjacency matrix of $\hat{G}$, then $\lambda \geq -2$. (iii) Is this true of false: $\hat{\hat{G}} = G$. (Justify your claim). [20]

(5) Let G be a graph with n vertices. Suppose λ_1 is the maximum eigenvalue of the Laplacian of G. Show that $\lambda_1 \leq n$. [15]

(6) Suppose $A = [a_{ij}]_{1 \leq i,j \leq n}$ is a $n \times n$ complex matrix. Then $|A|$ is defined as $(A^* A)^{\frac{1}{2}}$. Define $|A|^e$ (entry-wise modulus) by taking

$$|A|^e = [|a_{ij}|]_{1 \leq i,j \leq n}.$$

Show that in general $|A| \neq |A|^e \neq |(|A|^e)|$. [15]